

Magnetic field corrections to the repulsive Casimir effect at finite temperature

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I investigate the finite temperature Casimir effect for a charged and massless scalar field satisfying mixed (Dirichlet-Neumann) boundary conditions on a pair of plane parallel plates of infinite size. The effect of a uniform magnetic field, perpendicular to the plates, on the Helmholtz free energy and Casimir pressure is studied. The ζ -function regularization technique is used to obtain finite results. Simple analytic expressions are obtained for the zeta function and the free energy, in the limits of small plate distance, high temperature and strong magnetic field. The Casimir pressure is obtained in each of the three limits and the situation of a magnetic field present between and outside the plates, as well as that of a magnetic field present only between the plates is examined. It is discovered that, in the small plate distance and high temperature limits, the repulsive pressure is less when the magnetic field is present between the plates but not outside, than it is when the magnetic field is present between and outside the plates.

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I. INTRODUCTION

The attractive force between two conducting and electrically neutral parallel plates in a vacuum was first predicted theoretically by Casimir [1]. The first experimental evidence of the attractive Casimir force dates back more than fifty years [2], and many followed it. A comprehensive list of these experiments is available in the review article and book by Bordag et al. [3, 4].

Years later, the repulsive spherical Casimir effect was discovered by Boyer, who showed that a conducting and electrically neutral spherical shell in a vacuum is subject to an outward pressure caused by the quantum fluctuations of the electromagnetic field [5]. The repulsive Casimir effect for two parallel plates was also discovered by Boyer, within the framework of random electrodynamics, when he showed that two electrically neutral parallel plates in a vacuum, one perfectly conducting ($\epsilon \rightarrow 0$) and the other infinitely permeable ($\mu \rightarrow \infty$), are subject to a repulsive force [6]. This problem has been revisited more recently by several authors employing modern regularization methods, such as the zeta function technique [7, 8]. This technique has been used to calculate the finite temperature corrections to the repulsive Casimir effect within the framework of finite temperature field theory, in the case of a massless scalar field that mimics the electromagnetic field [9–11]. These authors impose Dirichlet boundary conditions for the scalar field on one plate and Neumann boundary conditions on the other plate, simulating a perfectly conducting and an infinitely permeable plate respectively, and assume this system to be in thermal equilibrium with a heat reservoir. The massless scalar field with mixed boundary conditions is equivalent to the electromagnetic system studied by Boyer in [5], and yields the same results if one accounts correctly for the two electromagnetic polarization states.

In this paper I will study the effect of a uniform magnetic field \vec{B} on the repulsive Casimir effect, by investigating a system very similar to the one studied in Refs. [9–11]; a massless, but charged, scalar field satisfying mixed boundary conditions on two plane parallel plates of infinite size and distance a , and in thermal equilibrium with a heat reservoir at temperature T . While several authors have investigated thermal and magnetic corrections to the attractive Casimir effect, see for example [12–14], a calculation of thermal and magnetic corrections to the repulsive Casimir effect associated with Boyer’s setup has not been done. Boyer’s pair of plates is the simplest system where we can observe the repulsive Casimir force at work. The spherical Casimir effect, also repulsive, requires very complex calculations for obtaining the Casimir force alone [15, 16], without including thermal or magnetic corrections. While thermal effects have been investigated within the context of the spherical geometry [17, 18], magnetic field effects have never been investigated. A solid understanding of magnetic and thermal effects for the Boyer setup will help understand the roles played by magnetic fields and temperature effects in other systems where the Casimir force is repulsive but the different geometry complicates the calculations significantly, such as the spherical system.

In Sec II I calculate the zeta function for this system, exact to all orders in eB , T and a , where e is the scalar field charge. In Sec. III I examine the small plate distance limit ($a^{-1} \gg T, \sqrt{eB}$) and obtain a simple analytic expression

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for the zeta function and the free energy. In Sec. IV I obtain simple analytic expressions for the zeta function and free energy in the high temperature limit ($T \gg a^{-1}, \sqrt{eB}$). In Sec. V I examine the strong magnetic field limit ($\sqrt{eB} \gg a^{-1}, T$) and derive simple analytic expressions for the zeta function and the free energy. In Sec. VI I obtain simple analytic expressions for the Casimir pressure in the small plate distance limit, the high temperature limit, and the strong magnetic field limit. I also examine the two different scenarios where a) the magnetic field is present between the plates as well as outside the plates, and b) where the magnetic field is only present between the plates, and obtain the Casimir pressure, in each of the three limits, for both scenarios. The conclusions with a discussion of the results of this work are presented in Sec. VII.

II. EVALUATION OF THE ZETA FUNCTION

For a system in thermal equilibrium with a heat reservoir, the imaginary time formalism of finite temperature field theory is convenient, and it allows only field configurations satisfying the following

$$\phi(x, y, z, \tau) = \phi(x, y, z, \tau + \beta), \quad (1)$$

for any τ , where $\beta = 1/T$ is the periodic length in the Euclidean time axis. In addition to the finite temperature boundary condition (1), I impose Dirichlet boundary conditions on one plate and Neumann boundary conditions on the other. The two plates are perpendicular to the z -axis and located at $z = 0$ and $z = a$. Dirichlet boundary conditions on the plate at $z = 0$, constrain the scalar field to vanish at that plate,

$$\phi(x, y, 0, \tau) = 0, \quad (2)$$

Neumann boundary conditions on the plate at $z = a$, constrain the derivative of the scalar field to vanish at that plate,

$$\frac{\partial \phi}{\partial z}(x, y, a, \tau) = 0. \quad (3)$$

In the slab region there is also a uniform magnetic field pointing in the z direction, $\vec{B} = (0, 0, B)$. The scalar field has charge e and interacts with the magnetic field.

The scalar field Helmholtz free energy is

$$F = \beta^{-1} \log \det (D_E | \mathcal{F}_a),$$

where the symbol \mathcal{F}_a indicates the set of eigenfunctions of the operator D_E which satisfy boundary conditions (1), (2) and (3), and D_E is defined as

$$D_E = \partial_\tau^2 + \partial_z^2 - (\vec{p} - e\vec{A})_\perp^2,$$

where \vec{A} is the electromagnetic vector potential, the subscript E indicates Euclidean time, and I use the notation $\vec{p}_\perp = (p_x, p_y, 0)$.

The zeta function technique allows me to use the eigenvalues of D_E to evaluate the free energy. The mixed boundary conditions (2) and (3) are satisfied only if the allowed values for the momentum in the z direction are

$$p_z = \frac{\pi}{a} \left(n + \frac{1}{2} \right), \quad (4)$$

where $n \in \{0, 1, 2, 3, \dots\}$. The eigenvalues of $(\vec{p} - e\vec{A})_\perp^2$ are the Landau levels

$$2eB \left(l + \frac{1}{2} \right), \quad (5)$$

with $l \in \{0, 1, 2, 3, \dots\}$. Using the eigenvalues (4) and (5), I construct the zeta function $\zeta(s)$ of D_E

$$\zeta(s) = L^2 \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \left(\frac{eB}{2\pi} \right) \sum_{l=0}^{\infty} \mu^{2s} \left[\frac{\pi^2}{a^2} \left(n + \frac{1}{2} \right)^2 + \frac{4\pi^2}{\beta^2} m^2 + eB (2l + 1) \right]^{-s},$$

where the factor $eB/2\pi$ takes into account the degeneracy per unit area of the Landau levels, L^2 is the area of the plates, and the parameter μ with dimension of mass keeps $\zeta(s)$ dimensionless for all values of s .

Once I will obtain a suitable closed form for the operator $\zeta(s)$, I will find the free energy by taking a simple derivative of $\zeta(s)$

$$F = -\beta^{-1}\zeta'(0). \quad (6)$$

With the help of the following identities

$$z^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-zt},$$

$$\sum_{l=0}^\infty e^{-(2l+1)z} = \frac{1}{2 \sinh z}, \quad (7)$$

where $\Gamma(s)$ is the Euler gamma function, I rewrite $\zeta(s)$ as

$$\zeta(s) = \frac{L^2 \mu^{2s}}{4\pi \Gamma(s)} \int_0^\infty dt t^{s-2} \frac{eBt}{\sinh eBt} \left(\sum_{n=0}^\infty e^{-\frac{\pi^2}{a^2} (n+\frac{1}{2})^2 t} \right) \left(\sum_{m=-\infty}^\infty e^{-\frac{4\pi^2}{\beta^2} m^2 t} \right). \quad (8)$$

It is not possible to evaluate (8) in closed form for arbitrary values of B , a and β , but it is possible to find simple expressions for $\zeta(s)$ when one or some of B , a and T are small or large. From these simple expressions of the zeta function, the free energy and the Casimir pressure will be obtained easily.

III. SMALL PLATE DISTANCE

I will first evaluate $\zeta(s)$ in the small plate distance limit ($a^{-1} \gg T, \sqrt{eB}$). In order to do that, I apply Poisson's resummation formula [19] to the m sum in (8) and find

$$\zeta(s) = \tilde{\zeta}_{B,a,T}(s) + \tilde{\zeta}_{B,a}(s), \quad (9)$$

where

$$\tilde{\zeta}_{B,a,T}(s) = \frac{L^2 \mu^{2s} \beta}{4\pi^{3/2} \Gamma(s)} \sum_{n=0}^\infty \sum_{m=1}^\infty \int_0^\infty dt t^{s-5/2} \frac{eBt}{\sinh eBt} e^{-m^2 \beta^2 / 4t} e^{-\frac{\pi^2}{a^2} (n+\frac{1}{2})^2 t}, \quad (10)$$

$$\tilde{\zeta}_{B,a}(s) = \frac{L^2 \mu^{2s} \beta}{8\pi^{3/2} \Gamma(s)} \sum_{n=0}^\infty \int_0^\infty dt t^{s-5/2} \frac{eBt}{\sinh eBt} e^{-\frac{\pi^2}{a^2} (n+\frac{1}{2})^2 t}. \quad (11)$$

The zeta function of Eq. (9) is equivalent to (8), but better suited for a small plate distance expansion.

After changing the integration variable from t to $tma\beta/\pi(2n+1)$ in (10), I obtain

$$\tilde{\zeta}_{B,a,T}(s) = \frac{L^2 \mu^{2s} \beta}{4\pi^{3/2} \Gamma(s)} \sum_{n=0}^\infty \sum_{m=1}^\infty \left[\frac{ma\beta}{\pi(2n+1)} \right]^{s-1/2} \int_0^\infty dt t^{s-5/2} \frac{eBt}{\sinh \left[\frac{eBtma\beta}{\pi(2n+1)} \right]} e^{-\pi(n+\frac{1}{2})m\beta(t+1/t)/2a}.$$

When $aT \ll 1$, only the term with $n=0$ and $m=1$ contributes significantly to the double sum so, using the saddle point method, I evaluate the integral and obtain

$$\tilde{\zeta}_{B,a,T}(s) = \frac{L^2 eB}{2\pi \Gamma(s)} \left(\frac{a\beta\mu^2}{\pi} \right)^s \frac{e^{-\pi\beta/2a}}{\sinh(\frac{eBa\beta}{\pi})}. \quad (12)$$

Next I evaluate (11) for $a\sqrt{eB} \ll 1$. In this case, I can set

$$\frac{eBt}{\sinh eBt} \approx 1 - \frac{1}{6}(eBt)^2 \quad (13)$$

and, after substituting (13) into (11), I integrate to find

$$\tilde{\zeta}_{B,a}(s) = \frac{\pi^{3/2} L^2 \beta}{8a^3 \Gamma(s)} \left(\frac{a\mu}{\pi} \right)^{2s} \left[\Gamma(s - \frac{3}{2}) \zeta_H(2s - 3, \frac{1}{2}) - \frac{e^2 B^2 a^4}{6\pi^4} \Gamma(s + \frac{1}{2}) \zeta_H(2s + 1, \frac{1}{2}) \right], \quad (14)$$

where

$$\zeta_H(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}$$

is the Hurwitz zeta function.

To calculate the free energy, it is sufficient to know $\zeta(s)$ for $s \rightarrow 0$. For small s I find

$$\frac{z^s}{\Gamma(s)} = s + \mathcal{O}(s^2), \quad (15)$$

$$z^{2s} \zeta_H(2s - 3, \frac{1}{2}) \frac{\Gamma(s - \frac{3}{2})}{\Gamma(s)} = -\frac{7\sqrt{\pi}}{720} s + \mathcal{O}(s^2), \quad (16)$$

and

$$z^{2s} \zeta_H(2s + 1, \frac{1}{2}) \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s)} = \frac{\sqrt{\pi}}{2} + \sqrt{\pi} [\gamma_E + \ln(2z)] s + \mathcal{O}(s^2), \quad (17)$$

where $\gamma_E = 0.5772$ is the Euler Mascheroni constant. Substituting (15), and (16) - (17) into (12) and (14) respectively, I obtain

$$\zeta(s) = -L^2 \beta \left[\frac{7\pi^2}{5,760a^3} + \frac{e^2 B^2 a}{48\pi^2} \left(\frac{1}{2s} + \gamma_E + \ln \frac{2a\mu}{\pi} \right) - \frac{eB}{2\pi\beta} \frac{e^{-\pi\beta/2a}}{\sinh(\frac{eBa\beta}{\pi})} \right] s,$$

valid in the limit of small plate distance and small s . The free energy in the small plate distance limit is obtained immediately using (6)

$$F = L^2 \left[\frac{7\pi^2}{5,760a^3} + \frac{e^2 B^2 a}{48\pi^2} \left(\gamma_E + \ln \frac{2a\mu}{\pi} \right) - \frac{eB}{2\pi\beta} \frac{e^{-\pi\beta/2a}}{\sinh(\frac{eBa\beta}{\pi})} \right]. \quad (18)$$

IV. HIGH TEMPERATURE

Next I evaluate $\zeta(s)$ in the high temperature limit ($T \gg a^{-1}, \sqrt{eB}$), and apply Poisson's resummation formula to the n sum in (8) to find

$$\zeta(s) = \zeta_{B,a,T}(s) + \zeta_{B,a}(s) + \zeta_{B,T}(s), \quad (19)$$

where

$$\zeta_{B,a,T}(s) = \frac{L^2 \mu^{2s} a}{2\pi^{3/2} \Gamma(s)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^n \int_0^{\infty} dt t^{s-5/2} \frac{eBt}{\sinh eBt} e^{-\frac{4\pi^2}{\beta^2} m^2 t} e^{-n^2 a^2/t}, \quad (20)$$

$$\zeta_{B,T}(s) = \frac{L^2 \mu^{2s} a}{8\pi^{3/2} \Gamma(s)} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dt t^{s-5/2} \frac{eBt}{\sinh eBt} e^{-\frac{4\pi^2}{\beta^2} m^2 t},$$

$$\zeta_{B,a}(s) = \frac{L^2 \mu^{2s} a}{4\pi^{3/2} \Gamma(s)} \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} dt t^{s-5/2} \frac{eBt}{\sinh eBt} e^{-n^2 a^2/t}.$$

Notice that $\zeta(s)$ from Eq. (19), while equivalent to (8) and (9), it is better suited for a high temperature expansion.

I evaluate (20) by changing the variable of integration ($t \rightarrow ta\beta n/2\pi m$), and obtain

$$\zeta_{B,a,T}(s) = \frac{L^2 \mu^{2s} a}{2\pi^{3/2} \Gamma(s)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^n \left(\frac{a\beta n}{2\pi m} \right)^{s-1/2} \int_0^{\infty} dt t^{s-5/2} \frac{eBt}{\sinh\left(\frac{eBta\beta n}{2\pi m}\right)} e^{-2\pi nma(t+1/t)/\beta}.$$

When $aT \gg 1$, all terms in the double sum are negligible when compared to the $m = n = 1$ term and, using the saddle point method, I find

$$\zeta_{B,a,T}(s) = -\frac{L^2 eB}{2\pi \Gamma(s)} \left(\frac{a\beta \mu^2}{2\pi} \right)^s \frac{e^{-4\pi a/\beta}}{\sinh(\frac{eBa\beta}{2\pi})}.$$

For $T \gg \sqrt{eB}$, I use (13), and find

$$\zeta_{B,T}(s) = \frac{L^2 \mu^{2s} a}{8\pi^{3/2} \Gamma(s)} \int_0^{\infty} dt t^{s-5/2} \left[\frac{eBt}{\sinh eBt} + 2 \sum_{m=1}^{\infty} \left(1 - \frac{e^2 B^2 t^2}{6} \right) e^{-\frac{4\pi^2}{\beta^2} m^2 t} \right],$$

and, after integration

$$\begin{aligned} \zeta_{B,T}(s) = \frac{L^2 a}{\Gamma(s)} & \left[\left(\frac{eB}{2\pi} \right)^{3/2} \left(\frac{\mu^2}{2eB} \right)^s \Gamma(s - \frac{1}{2}) \zeta_H(s - \frac{1}{2}, \frac{1}{2}) + \frac{2\pi^{3/2}}{\beta^3} \left(\frac{\mu\beta}{2\pi} \right)^{2s} \Gamma(s - \frac{3}{2}) \zeta_R(2s - 3) \right. \\ & \left. - \frac{e^2 B^2 \beta}{48\pi^{5/2}} \left(\frac{\mu\beta}{2\pi} \right)^{2s} \Gamma(s + \frac{1}{2}) \zeta_R(2s + 1) \right], \end{aligned}$$

where $\zeta_R(s)$ is the Riemann zeta function.

Last I calculate $\zeta_{B,a}(s)$, which contains only two of the parameters, B and a . In the high temperature limit, we need to explore the cases of $\sqrt{eB} \gg a^{-1}$ and of $a^{-1} \gg \sqrt{eB}$ separately. When $\sqrt{eB} \gg a^{-1}$, I can take

$$(\sinh eBt)^{-1} \approx 2e^{-eBt},$$

and write

$$\zeta_{B,a}(s) = \frac{L^2 \mu^{2s} a}{2\pi^{3/2} \Gamma(s)} \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} dt t^{s-5/2} eBt e^{-eBt} e^{-n^2 a^2/t},$$

which, after a change of the integration variable, becomes

$$\zeta_{B,a}(s) = \frac{L^2 \mu^{2s} a eB}{2\pi^{3/2} \Gamma(s)} \sum_{n=1}^{\infty} (-1)^n \left(\frac{an}{\sqrt{eB}} \right)^{s-1/2} \int_0^{\infty} dt t^{s-3/2} e^{-na\sqrt{eB}(t+1/t)}.$$

When $a\sqrt{eB} \gg 1$, only the term with $n = 1$ survives in the sum, and the integration is done quickly using the saddle point method to obtain

$$\zeta_{B,a}(s) = -\frac{L^2 eB}{2\pi \Gamma(s)} \left(\frac{\mu^2 a}{\sqrt{eB}} \right)^s e^{-2a\sqrt{eB}}.$$

In the case of high temperature and $a^{-1} \gg \sqrt{eB}$, I use (13), integrate and obtain

$$\zeta_{B,a}(s) = \frac{L^2 (\mu a)^{2s}}{4\pi^{3/2} a^2 \Gamma(s)} \left[(2^{2s-2} - 1) \zeta_R(3 - 2s) \Gamma(\frac{3}{2} - s) - \frac{e^2 B^2 a^4}{6} (2^{2s+2} - 1) \zeta_R(-1 - 2s) \Gamma(-\frac{1}{2} - s) \right].$$

The small s expansion of $\zeta(s)$, in the high temperature limit, is

$$\zeta(s) = \frac{L^2 a}{2\pi} \left[\frac{2\pi^3}{45\beta^3} + (eB)^{3/2} (\sqrt{2} - 1) \zeta_R(-\frac{1}{2}) - \frac{e^2 B^2 \beta}{6} \left(\frac{1}{2s} + \gamma_E + \ln \frac{\mu\beta}{4\pi} \right) - \frac{eB}{a} e^{-2a\sqrt{eB}} - \frac{eB e^{-4\pi a/\beta}}{a \sinh(\frac{eBa\beta}{2\pi})} \right] s,$$

for $\sqrt{eB} \gg a^{-1}$, and

$$\zeta(s) = \frac{L^2 a}{2\pi} \left[\frac{2\pi^3}{45\beta^3} + (eB)^{3/2}(\sqrt{2}-1)\zeta_R(-\tfrac{1}{2}) - \frac{e^2 B^2 \beta}{6} \left(\frac{1}{2s} + \gamma_E + \ln \frac{\mu\beta}{4\pi} \right) - \frac{3\zeta_R(3)}{16a^3} - \frac{e^2 B^2 a}{24} - \frac{eB e^{-4\pi a/\beta}}{a \sinh(\frac{eBa\beta}{2\pi})} \right] s,$$

for $a^{-1} \gg \sqrt{eB}$. The free energy in the high temperature limit is obtained using (6),

$$F = -\frac{L^2 a}{2\pi} \left[\frac{2\pi^3}{45\beta^4} + \frac{(eB)^{3/2}}{\beta}(\sqrt{2}-1)\zeta_R(-\tfrac{1}{2}) - \frac{e^2 B^2}{6} \left(\gamma_E + \ln \frac{\mu\beta}{4\pi} \right) - \frac{eB}{\beta a} e^{-2a\sqrt{eB}} - \frac{eB e^{-4\pi a/\beta}}{\beta a \sinh(\frac{eBa\beta}{2\pi})} \right] \quad (21)$$

for $\sqrt{eB} \gg a^{-1}$, and

$$F = -\frac{L^2 a}{2\pi} \left[\frac{2\pi^3}{45\beta^4} + \frac{(eB)^{3/2}}{\beta}(\sqrt{2}-1)\zeta_R(-\tfrac{1}{2}) - \frac{e^2 B^2}{6} \left(\frac{a}{4\beta} + \gamma_E + \ln \frac{\mu\beta}{4\pi} \right) - \frac{3\zeta_R(3)}{16\beta a^3} - \left(\frac{2\pi}{a^2 \beta^2} - \frac{e^2 B^2}{12\pi} \right) e^{-4\pi a/\beta} \right] \quad (22)$$

for $a^{-1} \gg \sqrt{eB}$, where I used a power series expansion of the hyperbolic sine because, for high temperature and $a^{-1} \gg \sqrt{eB}$, the quantity $eBa\beta \ll 1$.

V. STRONG MAGNETIC FIELD

In the strong magnetic field limit ($\sqrt{eB} \gg a^{-1}, T$), a form of the zeta function that can be easily expanded is obtained by Poisson-resumming over both m and n in (8)

$$\zeta(s) = \frac{L^2 a \beta \mu^{2s}}{16\pi^2 \Gamma(s)} \sum_{m,n=-\infty}^{\infty} (-1)^n \int_0^{\infty} dt t^{s-3} \frac{eBt}{\sinh eBt} e^{-\frac{\beta^2 m^2}{4t}} e^{-\frac{a^2 n^2}{t}},$$

and, using (7), is rewritten as

$$\zeta(s) = \frac{L^2 a \beta \mu^{2s}}{8\pi^2 \Gamma(s)} \sum_{m,n=-\infty}^{\infty} \sum_{l=0}^{\infty} (-1)^n eB \int_0^{\infty} dt t^{s-2} e^{-(2l+1)eBt} e^{-\frac{\beta^2 m^2}{4t}} e^{-\frac{a^2 n^2}{t}}.$$

After changing the integration variable, I obtain

$$\zeta(s) = \zeta_W(s) + \tilde{\zeta}(s),$$

with

$$\zeta_W(s) = \frac{L^2 a \beta \mu^{2s}}{8\pi^2 \Gamma(s)} \sum_{l=0}^{\infty} eB \int_0^{\infty} dt t^{s-2} e^{-(2l+1)eBt}, \quad (23)$$

and

$$\tilde{\zeta}(s) = \frac{L^2 a \beta \mu^{2s}}{8\pi^2 \Gamma(s)} \sum'_{m,n} \sum_{l=0}^{\infty} (-1)^n eB \left(\frac{\beta^2 m^2/4 + a^2 n^2}{(2l+1)eB} \right)^{(s-1)/2} \int_0^{\infty} dt t^{s-2} e^{-(t+1/t)\sqrt{(2l+1)eB(\frac{\beta^2 m^2}{4} + a^2 n^2)}}, \quad (24)$$

where the primed summation over n and m does not include the term with $n = m = 0$. Notice that $\zeta_W(s)$, as defined in Eq. (23), yields the zeta function of the one-loop Weisskopf effective Lagrangian for scalar QED [20], in the limit of a massless scalar. The integral present in (23) is evaluated easily to obtain

$$\zeta_W(s) = \frac{L^2 a \beta (eB)^2}{4\pi^2 \Gamma(s)} \left(\frac{\mu^2}{2eB} \right)^s \zeta_H(s-1, \tfrac{1}{2}) \Gamma(s-1). \quad (25)$$

Once I compare Eq. (25) to the well known result for the Weisskopf Lagrangian [20], I realize that I must take the arbitrary parameter $\mu = m_\phi$, where m_ϕ is the small mass of the scalar field, negligible when compared to the three relevant physical quantities a , T and \sqrt{eB} .

The integral in (24) is done using the saddle point method, to obtain

$$\tilde{\zeta}(s) = \frac{L^2 a \beta \mu^{2s}}{8\pi^{3/2} \Gamma(s)} \sum'_{m,n} \sum_{l=0}^{\infty} (-1)^n eB \left(\frac{\beta^2 m^2/4 + a^2 n^2}{(2l+1)eB} \right)^{s/2} \frac{[(2l+1)eB]^{1/4}}{[\beta^2 m^2/4 + a^2 n^2]^{3/4}} e^{-\sqrt{(2l+1)eB(\beta^2 m^2/4 + a^2 n^2)}}. \quad (26)$$

Only terms with $l = 0, m = \pm 1, n = 0$; or $l = 1, m = \pm 1, n = 0$; or $l = 0, m = 0, n = \pm 1$ contribute significantly to the sum in Eq. (26), and I obtain

$$\tilde{\zeta}(s) = \frac{L^2 a \beta (eB)^{5/4}}{4\pi^{3/2} \Gamma(s)} \left[\left(\frac{\mu^2 \beta}{2\sqrt{eB}} \right)^s \left(\frac{2}{\beta} \right)^{3/2} e^{-\beta\sqrt{eB}} + \left(\frac{\mu^2 \beta}{2\sqrt{3eB}} \right)^s \left(\frac{2\sqrt{3}}{\beta} \right)^{3/2} e^{-\beta\sqrt{3eB}} - \left(\frac{\mu^2 a}{\sqrt{eB}} \right)^s \left(\frac{1}{a} \right)^{3/2} e^{-2a\sqrt{eB}} \right].$$

In the strong magnetic field limit, the small s expansion of $\zeta(s)$ is

$$\zeta(s) = \frac{L^2 a \beta (eB)^{5/4}}{4\pi^{3/2}} \left[\frac{(eB)^{3/4}}{24\sqrt{\pi}} \left(\ln \frac{eB}{3\mu^2} - \frac{1}{2} \right) + \left(\frac{2}{\beta} \right)^{3/2} e^{-\beta\sqrt{eB}} + \left(\frac{2\sqrt{3}}{\beta} \right)^{3/2} e^{-\beta\sqrt{3eB}} - \left(\frac{1}{a} \right)^{3/2} e^{-2a\sqrt{eB}} \right] s,$$

and therefore, using (6), I obtain the free energy in the strong magnetic field limit

$$F = -\frac{L^2 a (eB)^{5/4}}{4\pi^{3/2}} \left[\frac{(eB)^{3/4}}{24\sqrt{\pi}} \left(\ln \frac{eB}{3m_\phi^2} - \frac{1}{2} \right) + \left(\frac{2}{\beta} \right)^{3/2} e^{-\beta\sqrt{eB}} + \left(\frac{2\sqrt{3}}{\beta} \right)^{3/2} e^{-\beta\sqrt{3eB}} - \left(\frac{1}{a} \right)^{3/2} e^{-2a\sqrt{eB}} \right]. \quad (27)$$

Notice that the arbitrary parameter μ has been replaced by m_ϕ , and this replacement should also occur in Eqs. (18), (21), and (22), the other three expressions I obtained for the free energy.

VI. CASIMIR PRESSURE

The Casimir pressure on the plates is given by

$$P = -\frac{1}{L^2} \frac{\partial F}{\partial a}. \quad (28)$$

When calculating the pressure, one must specify the temperature and magnetic field present in the region between the plates and in the region outside the plates, since the Casimir pressure depends on the conditions of both regions. Terms in the free energy that are proportional to a are uniform energy density terms, and will not contribute to the pressure if the medium outside the plates is at the same temperature and with the same magnetic field as the medium between the plates. I will assume that the medium inside and outside the plates is at the same temperature, and will investigate the cases when the magnetic field is only present between the plates, and when it is present between and outside the plates.

For small a , I use (18) into (28) and find

$$P_1 = \frac{7\pi^2}{1,920a^4} - \frac{e^2 B^2}{48\pi^2} \left(1 + \gamma_E + \ln \frac{2am_\phi}{\pi} \right) + \frac{eB}{4a^2} \frac{e^{-\pi\beta/2a}}{\sinh(\frac{eBa\beta}{\pi})},$$

when there is magnetic field only between the plates, and

$$P_2 = \frac{7\pi^2}{1,920a^4} - \frac{e^2 B^2}{48\pi^2} \left(1 + \ln \frac{2am_\phi}{\pi} \right) + \frac{eB}{4a^2} \frac{e^{-\pi\beta/2a}}{\sinh(\frac{eBa\beta}{\pi})},$$

for a magnetic field present inside and outside the plates. In both cases I neglected some smaller terms that do not contribute significantly to the pressure. The pressure is repulsive in both cases, as expected. Notice that the presence of a magnetic field between the plates, but not outside, weakens the repulsive pressure, since $\Delta P = P_1 - P_2$ is

$$\Delta P = -\frac{\gamma_E}{48\pi^2} e^2 B^2.$$

In the limit of high temperature and $\sqrt{eB} \gg a^{-1}$, I use (21) and (28) to find

$$P_1 = \frac{(eB)^{3/2}}{2\pi\beta}(\sqrt{2}-1)\zeta_R(-\tfrac{1}{2}) - \frac{e^2 B^2}{12\pi} \left(\gamma_E + \ln \frac{m_\phi \beta}{4\pi} \right) + \frac{(eB)^{3/2}}{\pi\beta} e^{-2a\sqrt{eB}} + 2 \frac{eB e^{-4\pi a/\beta}}{\beta^2 \sinh(\frac{eBa\beta}{2\pi})},$$

for the case when the magnetic field is present only between the plates, and

$$P_2 = \frac{(eB)^{3/2}}{\pi\beta} e^{-2a\sqrt{eB}} + 2 \frac{eB e^{-4\pi a/\beta}}{\beta^2 \sinh(\frac{eBa\beta}{2\pi})},$$

when the magnetic field is present between the plates and outside. I use (22), (28) and obtain the high temperature limit in the case of $a^{-1} \gg \sqrt{eB}$,

$$P_1 = \frac{(eB)^{3/2}}{2\pi\beta}(\sqrt{2}-1)\zeta_R(-\tfrac{1}{2}) - \frac{e^2 B^2}{12\pi} \left(\frac{a}{2\beta} + \gamma_E + \ln \frac{m_\phi \beta}{4\pi} \right) + \frac{3\zeta_R(3)}{16\pi\beta a^3} + \left(\frac{4\pi}{a\beta^3} - \frac{e^2 B^2 a}{6\pi\beta} \right) e^{-4\pi a/\beta},$$

for a magnetic field present only between the plates, and

$$P_2 = -\frac{e^2 B^2 a}{24\pi\beta} + \frac{3\zeta_R(3)}{16\pi\beta a^3} + \left(\frac{4\pi}{a\beta^3} - \frac{e^2 B^2 a}{6\pi\beta} \right) e^{-4\pi a/\beta},$$

when a magnetic field is present between the plates and outside. In both cases I neglected some smaller terms that do not contribute significantly to the pressure. Notice that the dominant term of the high temperature free energy (21) and (22) is the Stefan-Boltzman term, $\frac{\pi^2}{90\beta^4}$, but it does not contribute to the pressure because it is a uniform energy density term dependent on the temperature only. In the high temperature limit

$$\Delta P = \frac{(eB)^{3/2}}{2\pi\beta}(\sqrt{2}-1)\zeta_R(-\tfrac{1}{2}) - \frac{e^2 B^2}{12\pi} \left(\gamma_E + \ln \frac{m_\phi \beta}{4\pi} \right), \quad (29)$$

for both $\sqrt{eB} \gg a^{-1}$ and $a^{-1} \gg \sqrt{eB}$. Since $\zeta_R(-\frac{1}{2})$ is negative, the presence of a magnetic field between the plates, but not outside, weakens the repulsive pressure also in the high temperature limit, and can even reverse the pressure to an attractive one.

Finally, in the strong magnetic field limit I use (27), (28) and find

$$P_1 = \frac{(eB)^2}{96\pi^2} \left(\ln \frac{eB}{3m_\phi^2} - \frac{1}{2} \right) + \frac{(eB)^{5/4}}{\sqrt{2}(\pi\beta)^{3/2}} e^{-\beta\sqrt{eB}} + \frac{(3eB)^{5/4}}{\sqrt{2}(\pi\beta)^{3/2}} e^{-\beta\sqrt{3eB}} + \frac{(eB)^{7/4}}{2\pi^{3/2}\sqrt{a}} e^{-2a\sqrt{eB}},$$

$$P_2 = \frac{(eB)^{7/4}}{2\pi^{3/2}\sqrt{a}} e^{-2a\sqrt{eB}},$$

and

$$\Delta P = \frac{(eB)^2}{96\pi^2} \left(\ln \frac{eB}{3m_\phi^2} - \frac{1}{2} \right) + \frac{(eB)^{5/4}}{\sqrt{2}(\pi\beta)^{3/2}} e^{-\beta\sqrt{eB}} + \frac{(3eB)^{5/4}}{\sqrt{2}(\pi\beta)^{3/2}} e^{-\beta\sqrt{3eB}},$$

where the second and third term, exponentially suppressed for $\sqrt{eB}\beta \gg 1$, are in most cases much smaller than the first term. The dominant first term, and thus ΔP , can be positive or negative depending on the value of the strong magnetic field B . Therefore the presence of a magnetic field between the plates but not outside, will increase the repulsive pressure in the case of a very strong magnetic field, and will decrease it in the case of a moderately strong field.

VII. DISCUSSION AND CONCLUSIONS

In this paper I used the zeta function technique to study magnetic effects on the repulsive Casimir effect at finite temperature. I investigated a massless and charged scalar field satisfying mixed boundary conditions on two parallel plates, in thermal equilibrium with a heat reservoir and in the presence of a uniform magnetic field perpendicular to

the plates. I obtained simple analytic expressions for the Helmholtz free energy in the case of small plate distance (18), high temperature (21) and (22), and strong magnetic field (27).

Using the method described in Refs. [14, 22] and based upon the fact that, for a well behaved function $G(s)$, the derivative of $G(s)/\Gamma(s)$ at $s = 0$ is simply $G(0)$, I obtained exact expressions of the free energy suitable for accurate numerical evaluation in each of the three limits. I compared the values of the free energy obtained from my simple analytic expressions to the exact numerical values. In the small plate distance case I find that, for $aT \leq \frac{1}{2}$ and $a\sqrt{eB} \leq \frac{1}{2}$, Eq. (18) is within 3.3 percent of the exact value of the free energy, while for $aT \leq \frac{1}{5}$ and $a\sqrt{eB} \leq \frac{1}{5}$, Eq. (18) is within 0.001 percent of the exact free energy, showing that the discrepancy with the exact value falls very rapidly as aT and $a\sqrt{eB}$ decrease. The analytic expressions (21) and (22) for the high temperature limit of the free energy are considerably more accurate, since for $Ta \geq 2$ and $T/\sqrt{eB} \geq 2$ their discrepancy with the exact value of the free energy is less than 0.1 percent, and falls very rapidly as Ta and T/\sqrt{eB} grow. Finally, in the strong magnetic field limit, the free energy of Eq. (27) is within 0.2 percent of the exact value for $\sqrt{eB}a \geq 2$ and $\sqrt{eB}/T \geq 2$, with an accuracy similar to that of Eqs. (21) and (22).

The simple analytic expressions of the Casimir pressure in the three limits, obtained in Sec. VI, are as accurate as those of the free energy. It is discovered that, in the small plate distance and high temperature limit, the repulsive Casimir pressure is always weaker when a magnetic field is present between the plates but not outside, than it is when a magnetic field is present between the plates and outside. In the strong magnetic field limit this effect happens only for a moderately strong field, while the repulsion increases when a very strong magnetic field is examined.

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